Changing the single-particle spectrum, due to the interaction, is determined by the value $\Delta_{\vec{k}}$ that is the root of the equation (2.22).

Let us turn to the study of the equation. It has a trivial solution $\Delta_{\vec{k}} = 0$ corresponding to the normal state. Consider other options for the simplest case

$$U_{\vec{k}\vec{k}'} = \begin{cases} -\lambda & if \quad k_F - q < k, k' < k_F + q \\ 0 & beyond\ the\ diapazon \end{cases} \tag{2.25}$$

In this case, it follows from (2.22) within specified interval the value $\Delta_{\vec{k}}$ is also constant ($\Delta_{\vec{k}} = \Delta$), and the equation (2.22) looks as follows:

$$1 = \frac{\lambda}{2} \sum_{\vec{k}} \frac{1}{\sqrt{\Delta^2 + \xi_{\vec{k}}^2}} \tag{2.26}$$

We replace the sum with an integral by the rule $\sum_{\vec{k}} ... = (2\pi)^{-3} \int ... d^3k$. Believing chemical potential equal $\mu = E_F = \frac{\hbar^2 k_F^2}{2m}$, we will obtain

$$\xi_{\vec{k}} = \frac{\hbar^2 (k^2 - k_F^2)}{2m} \approx \hbar^2 k_F \underbrace{(k - k_F)}_{\delta} / m$$

Further we substitute $d^3\vec{k} = 4\pi k_F^2 dk$, and the equality (2.26) takes the form:

$$1 = \frac{\lambda k_F^2}{4\pi} \int_{-q}^{q} \left[ \Delta^2 + \left( \frac{\hbar^2 k_F}{m} \delta \right)^2 \right]^{-1/2} d\delta \tag{2.27}$$

Calculating integral and resolving the equation for $\Delta$, we will obtain

$$\Delta = \frac{2\hbar^2 k_F q}{m(\exp(1/\lambda N(E_F)) - 1)} \tag{2.28}$$

where $N(E_F) = \frac{m k_F}{2\pi^2 \hbar^2}$ is the density of electronic states at the Fermi level (without taking into account a spin).

The maximum change in the electron wave vector corresponding to the maximum (Debye) frequency of the virtual phonon $\omega_D$: $q = \omega_D / v_F$. In the weak-coupling approximation ($\lambda \cdot N(E_F) << 1$) we finally obtain

$$\Delta(0) = 2\hbar\omega_D \exp(-1/\lambda N(E_F)) \tag{2.29}$$

It becomes clear why the theory of superconductivity could not be created on the basis of perturbation theory for the accounting of interaction. The perturbation theory gives the corrections to the energy in the form of degrees of low interaction energy $\lambda$, and the obtained value $\Delta$ tends to zero as $\exp(-1/\lambda N(E_F))$, and for small values of $\lambda$ can not be expanded in a power series.

To clarify the physical meaning of $\Delta$ we will find the ground-state energy $E_0$. Substituting (2.20) and (2.21) to (2.16), we obtain

$$E_0 = \sum_{\vec{k}} \frac{\xi_{\vec{k}}(\sqrt{\xi_{\vec{k}}^2 + \Delta_{\vec{k}}^2} - \xi_{\vec{k}}) - \Delta_{\vec{k}}^2 / 2}{\sqrt{\xi_{\vec{k}}^2 + \Delta_{\vec{k}}^2}} \qquad (2.30)$$

In the case of the trivial solution $\Delta_{\vec{k}} = 0$ corresponding to the normal state we have $E_0 = 0$. If, however, $\Delta_{\vec{k}} \neq 0$, then $E_0 < 0$. Thus, this solution is energetically more favorable than the normal state. Replacing the sum by an integral and calculating it, we find that the energy decreases by $N(E_F)\Delta^2 / 2$.

When $\Delta_{\vec{k}} \neq 0$ the functions $u_{\vec{k}}$ and $v_{\vec{k}}$ are both different from zero, therefore, the new Fermi operators $A_0^+$ and $A_1^+$ correspond to the creation of new elementary excitations (quasi-particles) each of which is a superposition of the electron and hole states. Values $u_{\vec{k}}$ and $v_{\vec{k}}$ characterize probabilities of different states: $u_{\vec{k}}^2$ is a probability that when the excitations are absent the electron states with $\vec{k}$ and $-\vec{k}$ are not occupied simultaneously, and $v_{\vec{k}}^2 = \frac{1}{2}\left[1 - \frac{\xi_{\vec{k}}}{\sqrt{\Delta_{\vec{k}}^2 + \xi_{\vec{k}}^2}}\right]$ - that both are occupied.

It allows answering the question of how the electrons are distributed on the momentum and energy. Indeed, upon the transition from a normal state in superconducting in a sample there is the same quantity of electrons, but they don't fill Fermi sphere any more.

Figure 2.1 shows the graphs

a) the probability of filling the one-electron states with energy E in the normal metal,

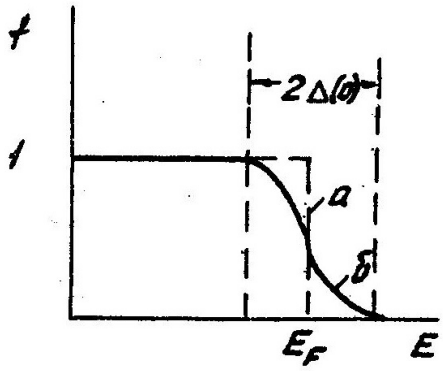


*Fig.2.1.*

b) $v_{\vec{k}}^2$ - the probability that the one-electron states $\vec{k}$ and $-\vec{k}$ are filled in the ground state of a superconductor.

The graph 2.1b shows that, as mentioned in §1.7, the one-electron states in the gap are occupied. The energy gap exists in the energy of elementary excitations, but not in the one-electron energy states.

We find the density of states in the spectrum of excitations of the superconductor, i.e., the number of states per unit energy. The states of excitements that existed in the normal metal, are reordered because of the gap. Since the quantity of states remains constant, we can write $N_n d\xi_{\vec{k}} = N_S d\varepsilon_{\vec{k}}$ , hence, using (2.24) and taking into account that the density of states in the normal metal near the Fermi level is constant, we obtain

$$N_S(\varepsilon) = N(E_F)\frac{\varepsilon}{\sqrt{\varepsilon^2 - \Delta^2}} \qquad (2.35)$$

This expression was used in plotting the density of states in the superconductor on Fig. 1.11.

These results explain the process of single-particle tunneling, shown in Figure 1.10.

But pairs of electrons can tunnel as well. On the scheme the superconducting electrons (which are parts of Cooper pairs) are located at zero level. After a break of a pair two excitements (two quasi-particles) are formed. All pairs form so-called "condensate" (no real condensate is present; this term is introduced only in the energy space). In fig. 2.2 possible processes of tunneling of pairs of electrons between two superconductors are represented. The first process is shown in fig. 2.2a: a pair of superconducting electrons leaves the left superconductor, without leaving excitations in it, and occupies quasi-particle states in the right one. In the second process (fig. 2.2b) two superconducting electrons leave the left sample, however two quasi-particles remain in it. Electrons tunnel into the right sample where drop out in condensate so in the right superconductor excitements don't arise.

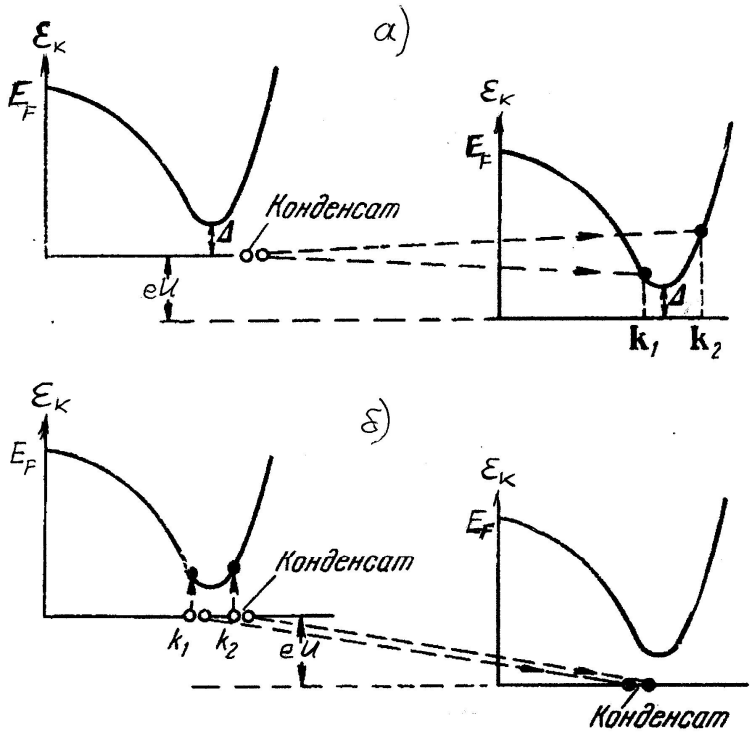


*Figure 2.2. Tunneling of electron pairs.*

As mentioned above, we have neglected the operator $H_2$, containing the product of four Fermi quasi-particle operators. This is acceptable for a small quantity of excitations. At nonzero temperatures the operator $H_2$ must be taken into account. That is why (2.29) indicates that the found value of $\Delta$ corresponds to $T = 0$. A rigorous calculation leads to expressions

$$2\xi_{\vec{k}} u_{\vec{k}} v_{\vec{k}} = (u_{\vec{k}}^2 - v_{\vec{k}}^2)\Delta \qquad (2.36)$$

$$\Delta = \lambda \sum_{\vec{k}'} u_{\vec{k}'} v_{\vec{k}'} (1 - n_{\vec{k}0} - n_{\vec{k}1}) \qquad (2.37)$$

Formula (2.37) shows that the size of the gap depends on the number of quasi-particles and their energy distribution.

Let us calculate the temperature dependence. Quasi-particles are distributed according to Fermi-Dirac law:

$$n_{\vec{k}} = \frac{1}{\exp(\varepsilon_{\vec{k}} / k_B T) + 1} \qquad (2.38)$$

Substituting (2.36) and (2.38) in (2.37) and passing to the integral, we obtain

$$1 = \frac{\lambda g(E_F)}{2} \int_0^{\hbar\omega_D} \frac{th\dfrac{\sqrt{\xi^2+\Delta^2}}{2k_BT}}{\sqrt{\xi^2+\Delta^2}} d\xi \qquad (2.39)$$

The dependence of half-width of an energy gap on temperature obtained as a result of calculations is given in fig. 2.3.

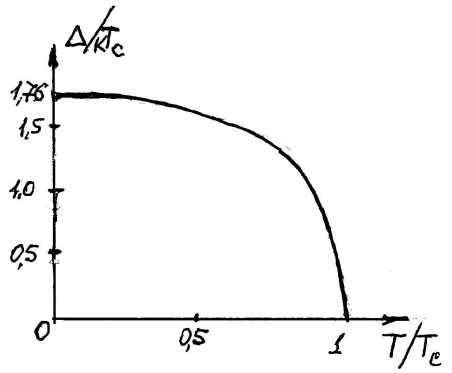


*Figure 2.3. The dependence $\Delta$ on temperature.*

We considered the situation when electrons interacted only through the exchange of virtual phonons, which provided attraction, and did not take into account the Coulomb repulsion. A rigorous calculation shows that it does not prevent very effectively the appearance of superconductivity. In particular, it can sometimes happen that even at resulting repulsion superconductivity, nevertheless, retained.

The explanation can be found, if we consider that the electron moves, leaving behind a positively charged "track", which disappears in a finite time.

**§2.2. Magnetic field inside superconductors**

The BCS theory explains the reasons for the restructuring of the energy spectrum of the metal, causing it to become superconducting. It is not directly involved in the calculation of the distribution of the fields and currents in superconducting sample, depending on its form and the character of the external magnetic field. In the following sections various theoretical descriptions considering a superconductor as the macroscopic environment are stated. In these descriptions the main equations turn out from a condition of a minimum of some thermodynamic potential. These equations allow calculating the coordinate dependence of magnetic fields, superconducting currents and a value of an energy gap.

The present paragraph contains some data on electrodynamics of continuous medium which will be used further. In particular, concepts of macroscopic characteristics of magnetic field are introduced, and the possibility of use of various thermodynamic potentials is justified.

As the characteristics of the magnetic field in a superconductor we can take microscopic field strength $\vec{h}$ at each point. In a steady state this field satisfies to Maxwell equations $div\vec{h} = 0$ and $rot\,\vec{h} = \vec{j}_S$ where $\vec{j}_S$ is a superconducting current density. When the currents are macroscopic quantities (for example the Meissner state), we can allocate them as separate sources of the field, while neglecting the diamagnetism of the substance, i.e., considering the magnetic permeability of the medium equal to unity. In this case, the magnetic induction is linked to the strength of the microscopic field by the relation $\vec{B} = \mu_0 \vec{h}$.

In those situations where there are microscopic currents and the field varies considerably over short distances, it makes sense to give up the microscopic analysis and move on to the macroscopic characteristics of the field. These are the macroscopic induction $\vec{B}$ and macroscopic field strength $\vec{H}$. Vector of magnetic induction in the medium is defined as the average by the volume (smaller than the typical sample size, but larger than the characteristic length of the field changes) of the field strength multiplied by the magnetic permeability of vacuum: $\vec{B} = \mu_0 \left\langle \vec{h} \right\rangle$. In points where there are no external sources of a field (a wire, a coil, etc.), the vector satisfies to the equations

$$div\,\vec{B} = 0, \quad rot\,\vec{B} = \mu_0 \left\langle \vec{j}_S \right\rangle, \tag{2.40}$$

where $\left\langle \vec{j}_S \right\rangle$ is the average density of the superconducting current.

We define the vector of macroscopic field strength $\vec{H}$ by the relation

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}), \tag{2.41}$$

where $\vec{M}$ is the magnetization vector equal to the magnetic moment per unit volume of the sample. Then the vector of field strength $\vec{H}$ satisfies to the equation

$$rot\,\vec{H} = 0. \tag{2.42}$$

It can be shown that the field strength $\vec{H}$ can be found as a partial derivative of the free energy density on magnetic induction:

$$H = \frac{\partial F}{\partial B} \tag{2.43}$$

In a vacuum, we have the relations

$$\vec{H} = \vec{h}, \quad \vec{B} = \mu_0 \vec{H} = \mu_0 \vec{h} \tag{2.44}$$

From (2.42) and the first equation from (2.40) follows that on border of two various environments the following boundary conditions have to be satisfied

$$B_{1n} = B_{2n}, H_{1\tau} = H_{2\tau} \tag{2.45}$$

If the sample has a form of a long rod and placed in an external field $\vec{H}_e$ parallel to its axis (by the external field, we mean a field generated by the external source in the absence of a superconducting media), the magnetic field $\vec{H}$ near the surface outside of the sample equals the external. From the condition of continuity of the tangential components of the vector $\vec{H}$ (2.45) it follows that the field $\vec{H}$ inside the sample at the border is equal to $\vec{H}_e$ as well. Then from (2.42) it follows that everywhere in the sample the field $\vec{H}$ is uniform and equal to $\vec{H}_e$. It explains the use of the relation $\vec{B} = \mu_0 (\vec{H}_e + \vec{M})$ instead of (2.41) in the curves in Figures 1.7. and 1.8.

In thermodynamics, it is proved that the magnetic field distribution at the given external currents corresponds to the minimum not of the free energy *F*, but of the Gibbs thermodynamic potential *G*, related to the free energy by the formula

$$G = F - \int \vec{B}\vec{H} dV \qquad (2.46)$$

The integral in (2.46) is taken over the entire space, i.e. over the area outside the sample as well. But as shown above, in the case of samples infinite along the direction of the external field, the field strength at all points of the sample is equal to the external, i.e. the energy of the field in this region does not depend on the current distribution in the sample. Therefore, this part of the energy can be excluded from consideration in minimizing the Gibbs potential, thus the integration in (2.46) has to be carried out only over the sample volume.

We emphasize once again that the case when the sample has a form of a rod, endless along the direction of the external field, is special. As stated above, at this geometry a magnetic field strength at any point outside of the sample is equal to the external, i.e. to the field strength which would be at this point in the absence of the sample. Therefore, carrying out the integration in the Gibbs potential before its minimization we can restrict ourselves by the volume of the sample. Furthermore, as shown above, in this case the macroscopic magnetic field strength $\vec{H}$ at all points within the sample is also equal to the external field $\vec{H}_e$. In other words, the field is uniform throughout the space. This fact often simplifies the search for solutions. In further consideration we will deal most often with this particular geometry. In cases of other geometries the possibility of excluding from consideration of the integral over the region outside of the sample will be analyzed specifically. The same applies to the assertion of the homogeneity of the field.