§2.3. Equation of F. and G. London

In 1935 brothers F. and G. London showed that in a limit when all fields and currents are weak and slowly change in space, the condition of minimum of free energy leads to a simple relation between fields and currents.

Brothers London in their theory based on the "two-fluid" model of a superconductor a year earlier proposed by Gorter and Cazimir. This model assumes the existence in a superconductor of two types of electrons - "normal" with concentration $n_n(T)$ and "superconducting" with concentration $n_S(T)$. The total concentration of the conduction electrons is $n = n_n + n_S$. The concentration of superconducting electrons decreases with increasing temperature and vanishes at $T = T_C$. When $T \to 0$ it tends to the total concentration of electrons. The superconducting current is provided by not fading movement of superconducting electrons, while the normal ones behave in usual way.

We will consider a pure metal with a parabolic conduction band; the effective mass of electrons equal to m. Free energy is as follows:

$$F = \int F_S dV + E_k + E_m, \qquad (2.47)$$

where F_S is the energy of electrons per unit volume in the condensed state in system of rest, and E_k is the kinetic energy connected with persistent currents. The drift velocity of electrons $\vec{v}$ at the point $\vec{r}$ is associated with the current density $\vec{j}_S$

$$n_S e\vec{v}(\vec{r}) = \vec{j}_S(\vec{r}), \qquad (2.48)$$

where e – the electron charge, n_S – the concentration of "superconducting" electrons.

The kinetic energy can be written as:

$$E_k = \frac{1}{2}\int n_S m v^2 dV, \qquad (2.49)$$

where the integral is taken over the sample volume. Expression (2.49) is valid provided that the velocity $\vec{v}(\vec{r})$ is a slowly changing function of the coordinates.

The energy associated with a magnetic field $\vec{h}(\vec{r})$ is given by:

$$E_m = \frac{\mu_0}{2}\int h^2 dV. \qquad (2.50)$$

Thus we consider the magnetic permeability of the medium equal to 1, and take into account the superconducting currents explicitly.

Then the field $\vec{h}(\vec{r})$ is connected with current density by Maxwell equation:

$$rot\,\vec{h} = \vec{j}_S \qquad (2.51)$$

Using (2.49)-(2.51), we present the free energy in the form

$$F = F_0 + \frac{\mu_0}{2}\int (h^2 + \lambda_L^2 \left|rot\,\vec{h}\right|^2)dV \qquad (2.52)$$

where $F_0 = \int F_S dV$, and a quantity λ_L, having the dimension of length, is defined as follows:

$$\lambda_L = \sqrt{\frac{m}{\mu_0 n_S e^2}} \qquad (2.53)$$

Note that Londons believe that the current is carried by individual electrons. At T = 0 all the electrons are "superconducting" and $n_S = n$. In fact, the electrons are grouped in pairs, so the values m, e and n_S in (2.53) should be changed by $2m, 2e, n/2$. It is easily seen that the shape of the formula (2.53) will not change, only n_S will be replaced by n. For simple metals such as Al, Sn etc., in which the mass m is close to the mass of a free electron, we find $\lambda_L \approx 0.05$ microns.

Let us consider a sample, infinite in the direction of the applied magnetic field. As shown in §2.2, in this case it is possible to minimize the integral calculated over only the sample volume without taking into account the area outside it.

Strictly speaking, as mentioned above, at constant external current producing the field, it is necessary to minimize the Gibbs potential. However, it can be shown that in this situation the minimization of the free energy leads to the same result. Later we will see that the London equation can be obtained in a more general case of other considerations.

We minimize F from (2.52) with respect to the distribution of the field $\vec{h}(\vec{r})$. If we change the field $\vec{h}(\vec{r})$ by $\delta\vec{h}(\vec{r})$ the energy F will get the increment δF:

$$\delta F = \mu_0 \int (\vec{h}\cdot\delta\vec{h} + \lambda_L^2 rot\,\vec{h}\cdot rot\,\delta\vec{h})dV = \mu_0 \int (\vec{h} + \lambda_L^2 rot\,rot\,\vec{h})\cdot\delta\vec{h}dV \qquad (2.54)$$

(We have integrated the second term by parts, using the formula of vector analysis $div(\vec{a}\times\vec{b}) = \vec{b}\cdot rot\,\vec{a} - \vec{a}\cdot rot\vec{b}$ for $\vec{a} = \delta\vec{h}, \vec{b} = rot\vec{h}$). Consequently, the configuration of the field inside the sample, giving the free energy minimum, must satisfy the equation

$$\vec{h} + \lambda_L^2\, rot\,rot\,\vec{h} = 0 \qquad (2.55)$$

Using Maxwell's equation (2.51) we can write the equation (2.55) in the form

$$rot\,\vec{j} = -\frac{\vec{h}}{\lambda_L^2} \qquad (2.56)$$

Equation (2.55) or (2.56) is called the London equation. Together with Maxwell equation (2.51), it allows to find the distributions of the field and currents. From it, in particular, it follows that the field inside a superconductor can not be uniform: $\vec{h}(\vec{r}) \neq const$. Let us consider some more consequences of the London equation.

2.3.1. The Meissner effect

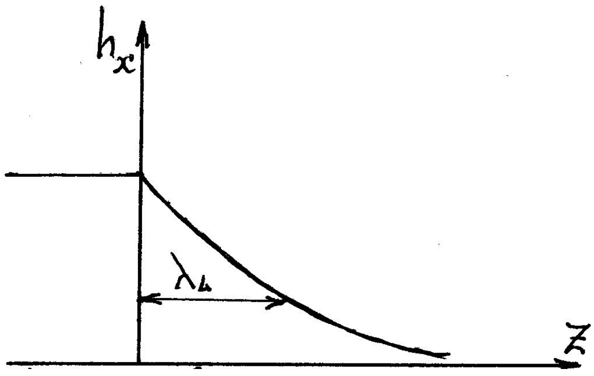
Now we apply the London equation (2.55) to the problem of the penetration of the magnetic field into a superconductor. Choose a simple geometry: the sample surface coincides with the plane xy, the region $z < 0$ is empty (Fig.2.4). Then the field $\vec{h}$ and the superconducting current density $\vec{j}_S$ depend only on z.

In addition to the equation (2.19), there are, as usual, the Maxwell equations

$$rot\,\vec{h} = \vec{j}_S \qquad (2.57)$$

$$div\,\vec{h} = 0 \qquad (2.58)$$

From the formula (2.56) it follows that $h_z = 0$ (as $\vec{j}$ is independent of *x* and *y*).



vacuum *superconductor*

Fig.2.4. Penetration of a weak magnetic field into the superconductor.

We choose the x axis along the direction of the field $\vec{h}$. Equation (2.58) is automatically satisfied, and the equation (2.57) shows that the current density $\vec{j}_S$ is directed along the axis *y*:

$$\frac{dh}{dz} = j_S \qquad (2.59)$$

Finally, equation (2.55) gives

$$\frac{d^2h}{dz^2} = \frac{h}{\lambda_L^2} \qquad (2.60)$$

The solution finite inside the superconductor is exponentially decreasing:

$$h(z) = h(0) \cdot \exp(-z/\lambda_L) \qquad (2.61)$$

i.e. the field penetrates into the superconductor only at a depth λ_L. This result obtained for the half-space can be easily generalized to the case of a sample of arbitrary shape. The depth of penetration λ_L is small, so it can be said that a weak magnetic field does not penetrate into the macroscopic sample, or in other words, the magnetic field is expelled from the sample. As it was mentioned earlier, this result was found in the experiments of Meissner and Ochsenfeld in 1933, before the creation of the London theory.

2.3.2. Thin films in the longitudinal magnetic field

Consider a sample in the shape of a thin film (Figure 2.5). It is infinite along the axes *x* and *y*, so that edge effects can be neglected. The field inside the sample is described by the equation:

$$\frac{d^2h}{dz^2} = \frac{h}{\lambda_L^2} \qquad (2.62)$$

Now, however, the boundary conditions are

$$h(d/2) = h(-d/2) = h_0 \qquad (2.63)$$

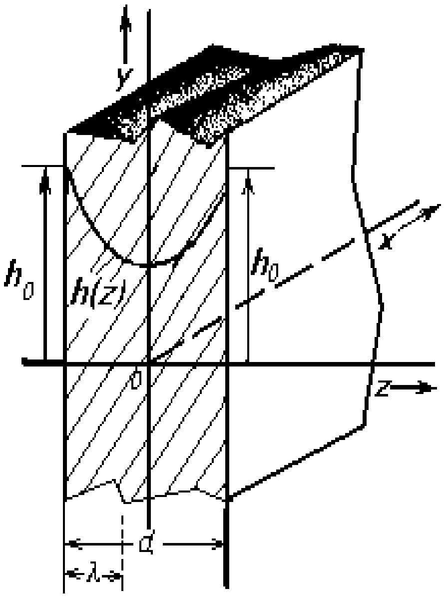


Figure 2.5. The distribution of the magnetic field inside the thin superconducting film.

The general solution can be written as:

$$h(z) = C_1 \cdot \exp(-z/\lambda_L) + C_2 \cdot \exp(z/\lambda_L) \qquad (2.64)$$

From the symmetry of the problem on z it follows that $C_1 = C_2$. Satisfying the boundary conditions (2.63), we finally obtain

$$h(z) = h_0 \frac{ch(z/\lambda_L)}{ch(d/2\lambda_L)} \qquad (2.65)$$

The dependence (2.65) is shown in Figure 2.5. By decreasing the film thickness d the efficiency of weakening of the field decreases. When $d << \lambda_L$ the field is almost uniformly permeates the superconducting film. The energy expended in ejecting the external field is small, and therefore, the external magnetic field required for the destruction of superconductivity, is more than the same value for a bulk superconductor.

§2.4. The Pippard equation

In deriving the London equation we assumed that the velocity $\vec{v}(\vec{r})$, or what is the same, the superconducting current density $\vec{j}(\vec{r})$, changes slowly in space. Let us specify what means "slowly."

In a condensed system the velocities of two electrons 1 and 2 are correlated, if the distance between them is less than a certain value R_{12}, we denote it ξ_0. Our conclusion is valid, if the velocity $\vec{v}(\vec{r})$ varies slightly over distances of the order of ξ_0. To evaluate ξ_0, note that the scope of the essential values of the momenta of the electrons is given by inequality

$$E_F - \Delta < \frac{p^2}{2m} < E_F + \Delta \qquad (2.66)$$

where E_F - the Fermi energy, Δ - the half-width of the energy gap.

The corresponding variation in momentum is $\Delta p = 2\Delta / v_F$ where $v_F = p_F / m$ - Fermi electron velocity. Because of the uncertainty relation we estimate the width of the corresponding wave packet $\delta x \approx \hbar / \Delta p$, based on which we can introduce the characteristic length called the coherence length of the superconductor (sometimes referred to as a size of pair):

$$\xi_0 = \frac{\hbar v_F}{\pi \Delta} . \qquad (2.67)$$

(Factor $1/\pi$ is introduced for reasons of convenience).

From London equation it follows that the characteristic length of the changes of the field, currents and velocity is the length λ_L, so this equation is valid only on the condition $\lambda_L >> \xi_0$.

In the simple (non-transition) metals the penetration depth λ_L is small (hundreds $\mathring{A}$). And the Fermi velocity is high (thousands of km/s), therefore the parameter ξ_0 is large (for Al).It means that the London equation is not suitable to describe the process in such metals. In this case, the London equation (2.56) has to be replaced by another one, a little bit more complicated. Its form was proposed by Pippard. We will call such superconductors the type-I superconductors. Those superconductors, to which the London equation is applicable, i.e., the condition $\lambda_L >> \xi_0$ is valid, are called type-II superconductors. We will discuss it in detail later.

Historically it happened, that in the 20 years after the discovery of the Meissner effect the experiments were carried out mainly on the type-I superconductors, and only later the study of type II superconductors began. It is interesting to remark that the theory developed in the reverse order: the theory of London, was established in 1935, and its modifications to the type-I superconductors were offered by Pippard not before 1953.

To complete the picture it is worth mentioning the superconducting alloys, in which the coherence length ξ_0 and the penetration depth λ_L depends on the mean free path of electrons. With its reduction ξ_0 decreases and λ_L increases. Therefore often it turns out that the addition of impurities into a type-I superconductor turns it to type-II superconductor.

Pippard idea can be explained as follows.

Using instead of the field $\vec{h}(\vec{r})$ vector potential $\vec{A}(\vec{r})$ associated with the field by

$$rot\,\vec{A} = \vec{B} = \mu_0 \vec{h} \qquad (2.68)$$

we write the London equation (2.20) $rot\,\vec{j} = -\vec{h}/\lambda_L^2$ in the form

$$\vec{j} = -\frac{\vec{A}}{\mu_0 \lambda_L^2} \qquad (2.69)$$

We note that (2.68) defines the vector potential $\vec{A}(\vec{r})$ ambiguously. In the derivation of (2.69), we chose it so that $div\,\vec{A} = 0$ (it is called the London calibration).

The relation (2.69) is applicable only when both $\vec{j}$ and $\vec{A}$ vary slowly in space. In general it can be assumed that the current density $\vec{j}(\vec{r})$ at a point $\vec{r}$ depend on the vector potential in all adjacent points $\vec{r}\,'$ satisfying the condition $\left|\vec{r}-\vec{r}'\right|<\xi_0$. Pippard proposed the following phenomenological relationship:

$$\vec{j}(\vec{r})=C\int\frac{\left(\vec{A}(\vec{r}')\cdot\vec{R}\right)\vec{R}}{R^4}\exp(-\frac{R}{\xi_0})dV',\ \text{where } \vec{R}=\vec{r}-\vec{r}' \qquad (2.70)$$

Later, on the basis of the microscopic BCS theory it has been shown that the exact relationship between the current and the field is very similar to the ratio (2.70), but the mathematical expression is much more complicated. Therefore, the approximate result of Pippard still has not lost its value.