§2.5. Ginzburg – Landau theory

2.5.1. Basic equations

London theory is applicable only to those states in which the concentration of the superconducting charge carriers is constant throughout the sample volume. In 1950, the soviet physicists V.L. Ginzburg and L.D. Landau published a theory that does not require constant concentration n_S. This theory comes from the fact that the transition from the normal (N) to the superconducting (S) state is a phase transition of type II, i.e. not accompanied by the release or absorption of heat.

The theory of such transitions was created by Landau earlier. It was attended by a parameter called the order parameter, which in the new phase (in this case - in the superconducting phase) is equal to zero at $T = T_C$ and increases with decreasing temperature. As such an order parameter in the superconductor Ginzburg and Landau considered the wave function $\psi(\vec{r})$, so that $|\psi(\vec{r})|^2$ equals the concentration of superconducting carriers.

Note that the conditions of applicability of the Ginzburg-Landau theory is the proximity of temperature of the sample to the critical temperature.

a) Let us first consider the simplest case when there is no magnetic field and the parameter ψ is independent of the coordinates. Since the value of ψ in the superconducting phase should gradually drop to zero when the temperature is approached T_C, the free energy F_S near T_C can be expanded in a power series in $|\psi|$:

$$F_S = F_N + \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + ... \qquad (2.71)$$

where F_N is a free energy in the normal state.

Let us discuss the expression (2.71).

At $T = T_C$, i.e. $\psi = 0$, the energy of the superconducting state is equal to the energy of normal, it explains the appearance of the member F_N in (2.71). The absence of a linear term in $|\psi|$ is due to symmetry considerations, we will not discuss this situation and just take it on faith.

We now show that the coefficient α should equal zero at $T = T_C$. Since at $T < T_C$ the energy of the superconducting state must be less than the energy of normal, the coefficient α should be negative. At $T > T_C$, i.e. across the transition, it should be positive. So, at the transition point it becomes zero and in the lowest order in ($T - T_C$) we obtain

$$\alpha = k(T - T_C) \qquad (2.72)$$

The coefficient k is positive.

The very point of transition must be stable, i.e. at $\alpha = 0$, the function $F(|\psi|)$ should have minimum at $|\psi| = 0$. Thus, the third order term must be equal to zero, and the fourth-order term should be positive. Hence it follows that the coefficient β is positive.

Near T_C we keep only two terms of the expansion. In this case the coefficients α and β are quite simply related to the thermodynamic critical field B_C and the equilibrium density of Cooper pairs at arbitrarily large distance from the border $n_S(\infty) = |\psi_\infty|^2$. Indeed, taking into account (1.2) $F_N - F_S = \mu_0 H_C^2/2$, we obtain the following relation

$$F_S - F_N = \alpha|\psi_\infty|^2 + \frac{1}{2}\beta|\psi_\infty|^4 = -\frac{1}{2\mu_0}B_C^2 \qquad (2.73)$$

The second equation for α and β is obtained from the condition of minimum F_S at equilibrium, i.e. $\frac{\partial F_S}{\partial(|\psi|^2)} = 0$, whence we have

$$\alpha + \beta|\psi_\infty|^2 = 0 \qquad (2.74)$$

By solving the system of equations (2.73), (2.74), we find

$$\alpha = -\frac{1}{\mu_0}\frac{B_C^2}{n_S(\infty)} \qquad (2.75)$$

$$\beta = \frac{1}{\mu_0}\frac{B_C^2}{n_S^2(\infty)} \qquad (2.76)$$

b) Now we assume that the order parameter varies slowly from point to point. One can show that in this case, the magnetic field effect on free energy F_S will manifest itself in the addition of two members: $\frac{1}{2m'}\left|(-i\hbar\vec{\nabla} - e'\vec{A})\psi\right|^2$ and $\frac{B^2}{2\mu_0}$, where $m' = 2m$ and $e' = 2e$ - the mass and epy charge of the particle, i.e. Cooper pair; $\vec{A}$ - the vector potential of the magnetic field. Note that creating their theory the authors, not knowing about the existence of Cooper pairs, thought that the charge carriers are electrons and believed $m' = m$ and $e' = e$.

The first of these members is usually obtained in quantum mechanics from the kinetic energy when replacing the momentum of a particle on the generalized momentum in the magnetic field. It describes the energy of the superconducting currents, as well as the energy associated with the spatial inhomogeneity of the distribution of the Cooper pairs. The second term corresponds to the energy of the magnetic field.

As mentioned in §2.2, the magnetic field distribution at the given external currents should be found of the minimum condition not for the free energy F, but for the Gibbs thermodynamic potential G. To find the value G_S of the sample one should subtract from the total value of potential $G = (F_S - BH)$ the potential of the external field $\frac{B_e^2}{2\mu_0} - B_e H_e = -\frac{B_e^2}{2\mu_0}$. Let the sample is a rod, endless along the external field (§2.2). In this case, the macroscopic magnetic field $\vec{H}$ is equal to the external field $\vec{H}_e$. Taking this into account, we get

$$G_S = F_N + \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{(B - B_e)^2}{2\mu_0} + \frac{1}{2m'}\left|(-i\hbar\vec{\nabla} - e'\vec{A})\psi\right|^2 \qquad (2.77)$$

The physical meaning of the term $\frac{(B-B_e)^2}{2\mu_0}$ is that it corresponds to the energy necessary for the magnetic field, which in the absence of the superconductor equals $\vec{B}_e$, accepts the current value $\vec{B}$. When $\vec{B}=0$ (Meissner phase) this member is equal to a full magnetic energy.

Gibbs potential of the sample is obtained by integrating (2.77) over the whole volume . Minimizing the resulting expression by ψ and $\vec{A}$ by means of variational method and taking into account (2.40), we obtain two equations of Ginzburg-Landau:

$$\alpha\psi+\beta|\psi|^2\psi+\frac{1}{2m'}\left|-i\hbar\vec{\nabla}-e'\vec{A}\right|^2\psi=0 \qquad (2.78)$$

$$\vec{j}_S=\frac{e'i\hbar}{2m'}(\psi\vec{\nabla}\psi^*-\psi^*\vec{\nabla}\psi)-\frac{e'^2}{m'}|\psi|^2\vec{A} \qquad (2.79)$$

where ψ^* is the complex conjugate function of ψ.

From the requirement of vanishing of the current component perpendicular to the border superconductor-vacuum (or insulator), we get the following boundary conditions for this system of equations

$$\vec{n}(-i\hbar\vec{\nabla}-e'\vec{A})\psi=0 \qquad (2.80)$$

where $\vec{n}$ is the normal to the boundary. In the case of the superconductor-normal metal the right side of (2.80) takes the form $i\lambda\psi$ where λ is a nonzero real constant.

Solving the system of equations (2.78) - (2.80), together with the Maxwell equations, one can find $\psi(\vec{r})$ and $\vec{j}(\vec{r})$, and then $\vec{B}(\vec{r})$.

We note that (2.79) is identical with the expression for the current density, introduced in quantum mechanics. Thus, the applicability of the second Ginzburg-Landau equation is not limited by a condition of closeness to the critical temperature.

2.5.2. Two characteristic lengths

We will show that the Ginzburg - Landau equations (2.78) - (2.79) contain two characteristic lengths.

a) First, consider the case where the magnetic fields and currents are absent. Select the calibration, in which the function $\psi(\vec{r})$ is real and for simplicity confine ourselves to one-dimensional case. Then (2.78) is noticeably simplified:

$$-\frac{\hbar^2}{2m'}\frac{d^2\psi}{dx^2}+\alpha\psi+\beta\psi^3=0 \qquad (2.81)$$

This equation has two obvious solutions: 1) a decision $\psi=0$ relating to a normal state; 2) a decision $\psi=\psi_0$, where $\psi_0^2=-\alpha/\beta$, describing the ordinary superconducting state. The second solution exists and corresponds to a lower energy when $\alpha<0$; i.e. when $T<T_C$. However, we would like to review the decisions of a more general type, such as when under the influence of an

external factor, the order parameter $\psi(x)$ at some point has a value other than ψ_0. How does $\psi(x)$ behave in the vicinity of this point?

Let us turn to the reduced variable $f=\psi/\psi_0$ and introduce the notation $\frac{\hbar^2}{2m'|\alpha|}=\xi^2(T)$ where the parameter $\xi(T)$ has the dimension of length. Equation (2.81) takes the form

$$-\xi^2(T)\frac{d^2 f}{dx^2}-f+f^3=0 \tag{2.82}$$

It follows that the parameter $\xi(T)$ is the natural unit of measurement of distance, at which the function f can change. We will call it the coherence length at a given temperature T. It can be shown that for pure metals

$$\xi(T)=0{,}74\left(1-\frac{T}{T_C}\right)^{-0{,}5}\xi_0 \ . \tag{2.83}$$

From this expression it is seen that, at temperatures near T_C, the order parameter varies little over distances of the order of the pair dimension ξ_0.

b) The second characteristic length appears when we turn to the consideration of electromagnetic effects, for example, when calculating the depth of penetration of the weak magnetic field.

If the external field is small, in the first order in the field h in the superconductor the parameter $|\psi|^2$ can be replaced by its equilibrium value $|\psi_0|^2$ in the absence of the field. Taking the curl of both sides of the second Ginzburg-Landau equation (2.79) and taking into account that $rot\,\vec{h}=\vec{j}_S$ and $rot\,\vec{A}=\mu_0\vec{h}$ we arrive at the equation of London type (because ψ_0 does not depend on the coordinates)

$$rot\,\vec{j}=-\frac{e'^2}{m'}\psi_0^2\mu_0\vec{h} \tag{2.84}$$

Comparing (2.84) with the London equation (2.56), we obtain an expression for the characteristic length

$$\lambda(T)=\sqrt{\frac{m'}{\mu_0 n_S e'^2}} \tag{2.85}$$

Note that the depth λ is proportional to ψ_0^{-1}, i.e. $(1-T/T_C)^{-0{,}5}$. For pure metal BCS theory gives

$$\lambda(T)=\frac{1}{\sqrt{2}}\left(1-\frac{T}{T_C}\right)^{-0{,}5}\lambda_L(0), \tag{2.86}$$

where $\lambda_L(0)=\sqrt{\frac{m}{\mu_0 ne^2}}$ (2.87) - London penetration depth at $T=0$.

Since, as mentioned above, the applicability of the second Ginzburg-Landau equation is not limited to a condition of closeness to the critical temperature, the obtaining of (2.84) and (2.85) can be regarded as an independent derivation of London equation, the applicability of which is determined only by the condition of independence of the module of order parameter on coordinates. This result is more general than that obtained in §2.3.

c) We have found two characteristic lengths, $\xi(T)$ and $\lambda(T)$, determining the behavior of a superconductor near the critical temperature. Both values are proportional to $(T_C - T)^{-0,5}$, so their ratio $\kappa = \dfrac{\lambda(T)}{\xi(T)}$, which is called the Ginzburg-Landau parameter of the substance, is of special interest.

Depending on the value of κ the superconductors are divided into two types:

$\kappa < 1$ (i.e. $\lambda < \xi$) - type I superconductors;

$\kappa > 1$ (i.e. $\lambda > \xi$) - type II superconductors.

Later we will see that the exact boundary corresponds to the value $\kappa = 1/\sqrt{2}$.

2.5.3. Problems with constant amplitude of the order parameter

We now use the Ginzburg-Landau equations to solve some specific problems. First, consider the simplest case when the amplitude of the order parameter $|\psi|$ is the same at all points of the sample. We have already encountered with this situation at the calculation of the depth of penetration of the weak magnetic field into the bulk sample. Now we'll investigate the case of another kind. We consider thin samples (film, wire, etc.), in which arbitrary change of ψ over the thickness is disadvantageous, since it would lead to a sharp increase of a member $|\nabla\psi|^2$ in the expression for the Gibbs potential. The field $\vec{h}$ and current density $\vec{j}$ are not assumed to be weak, so the amplitude $|\psi|$ remains constant, but not necessarily equal to its unperturbed value $|\psi_0|$.

2.5.3.1. The critical current in a thin film

Let us consider a current with density $\vec{j}$ flowing along the x axis in a film of thickness d, as shown in Fig. 2.6a.

Let a film be thin, i.e. $d << \xi(T)$ and $d << \lambda(T)$. The first inequality provides a constant amplitude over the film thickness, and the second one - the constancy of the current density. The equations are considerably simplified.

Indeed, we put

$$\psi = |\psi| \exp(i\varphi(\vec{r})), \qquad (2.88)$$

where the amplitude $|\psi|$ does not depend on $\vec{r}$.

The expression for the current density (2.79) can be written as

$$j = \frac{e'}{m'}|\psi|^2(\hbar\frac{\partial\varphi}{\partial x} - e' A_x) = e'|\psi|^2 v, \qquad (2.89)$$

where

$$v = \frac{1}{m'}(\hbar\frac{\partial\varphi}{\partial x} - e' A_x) \qquad (2.90)$$

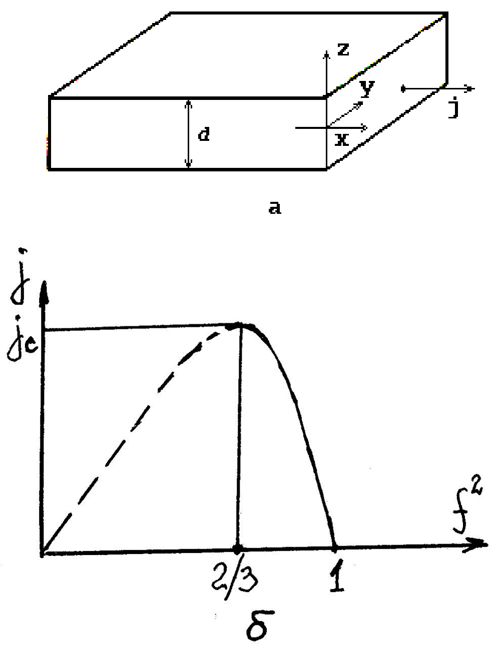

Figure 2.6. The calculation of the critical current of the thin film.

The quantity v - is the velocity of "particles" in the state with a wave function ψ .
Gibbs potential (2.77) also takes the simple form

$$G_S = F_N + |\psi|^2(\alpha + \frac{1}{2}\beta|\psi|^2 + \frac{1}{2}m'v^2) + \frac{(B - B_e)^2}{2\mu_0} \qquad (2.91)$$

From the condition of minimum of (2.91) relative to $|\psi|^2$ we obtain

$$(\alpha + \beta|\psi|^2 + \frac{1}{2}m'v^2) = 0 \qquad (2.92)$$

Assuming $|\psi| = \psi_0 \cdot f$, where $\psi_0^2 = -\frac{\alpha}{\beta}$, and excluding the velocity v from (2.89) and (2.92), we obtain

$$j = e'\psi_0^2\left(\frac{2|\alpha|}{m'}\right)^{1/2} f^2(1-f^2)^{1/2} = e'\psi_0^2\frac{\hbar}{m'\xi(T)}f^2(1-f^2)^{1/2} \qquad (2.93)$$

The relationship between j and f^2 is shown in ris.2.6b. When j increases from zero the function f decreases from the initial value of 1 to 0.8 at $j = j_C = 2e'\psi_0^2\frac{\hbar}{3\sqrt{3}m'\xi(T)}$. When $j > j_c$ there are no solutions, that is, the film is in the normal state. At the transition point the parameter f jumps from 0.8 to 0. The value j_c is called the critical current density of the film.

From a physical point of view, the existence of the critical current density can be easily explained. The current creates a magnetic field that penetrates the sample. At a certain value of current density the magnetic field at some points exceeds the critical value and the sample can no longer remain superconducting in the entire volume. In the future, the critical currents in superconductors will be discussed in detail.