**§3.3. The magnetic properties of type II superconductors.**

From the foregoing discussion it follows that in the case $\xi < \lambda$ the creation of the interface must be connected with the energy gain. Therefore, we should expect that magnetic field can penetrate into a superconductor already in the field $H < H_C$, while there are irregularities in the spatial distribution of the magnetic field and the density of Cooper pairs.

It turns out that the condition $\xi < \lambda$ can be fulfilled in any case if we reduce the mean free path of electrons. The fact is that when it decreases the penetration depth $\lambda$ slightly increases, and the coherence length $\xi$ rapidly decreases. The mean free path can be easily reduced by doping a superconductor with foreign metals. The electrons are scattered by the atoms of impurities and their mean free path is reduced.

Superconductors with $\xi < \lambda$ are called type-II superconductors. They are characterized by the following macroscopic properties.

1. In the cylinder placed in a longitudinal magnetic field, the Meissner effect occurs to a value $H_{C1}$ being significantly less than $H_C$.

2. When $H > H_{C1}$ the lines of force penetrate the sample, but only partially. This is the case for the field $H_{C1} < H < H_{C2}$. The field $H_{C2}$ is higher than $H_C$ and in some cases is very high.

3. When $H > H_{C2}$ a macroscopic sample does not push out the flow. However, even in this case, the superconductivity is not completely destroyed: in the field region $H_{C2} < H < H_{C3}$ on the cylinder surface there remains a superconducting layer with a thickness of the order less than a micron. The physical reason for the presence of such layer is as follows: a small superconducting region can easier be formed near the sample surface like an air bubble is easier formed on the bottom of the glass of lemonade than anywhere inside.

Changing of the fields $H_{C1}, H_{C2}, H_{C3}$ with temperature is shown in Figure 3.8.

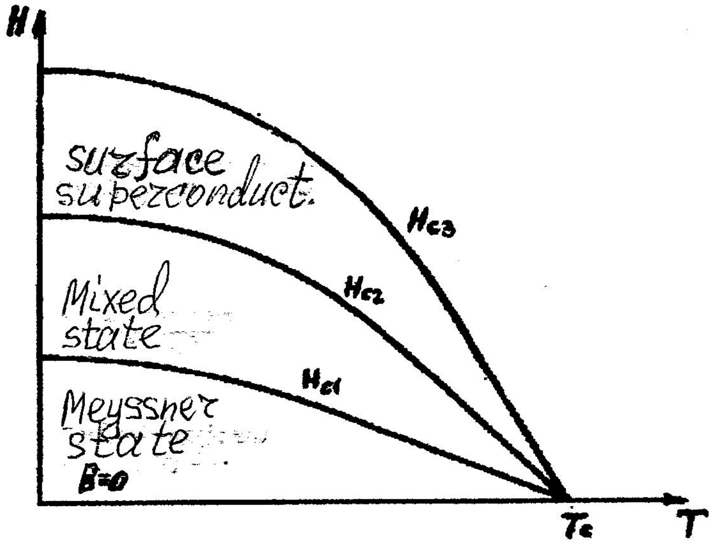


*Figure 3.8. The phase diagram for a type-II superconductor in the form of a long cylinder.*

Consider a range of fields $H_{C1} < H < H_{C2}$ in which the flux penetrates partially. For the first time he existence of such a region has been demonstrated by Shubnikov in 1937. Therefore, sometimes it is called the Shubnikov phase. We will call this area the vortex or the mixed state.

Typical plots $B(H)$ for type I and type II superconductors are shown in Figure 3.9. As it was mentioned earlier, the behavior of the sample in a magnetic field is often described by a magnetization curve, that is, the dependence of the magnetization $M$ on the external magnetic field $H_e$. In Figure 3.10 such graphs derived from the Figure 3.9 are shown. Note that if the values $H_C$ are the same for both materials, then the shaded area of curvilinear triangles on Figure 3.10 are equal.

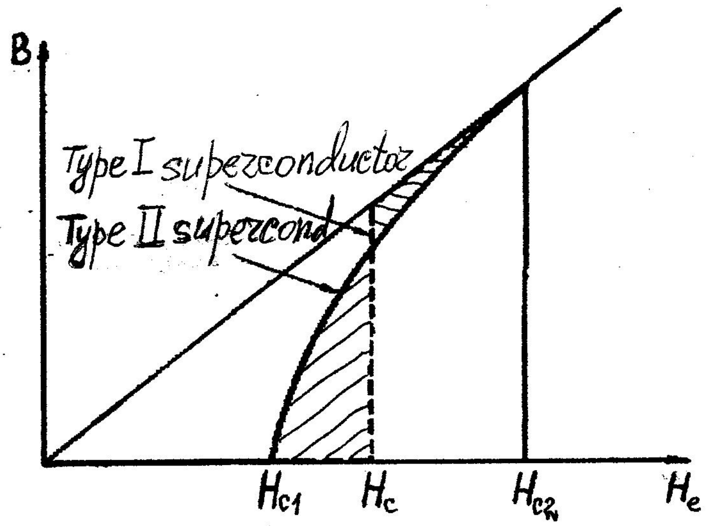


*Figure 3.9. The dependence of the induction $B$ on the applied field $H_e$ for type I and II superconductors in the form of a long cylinder.*

As mentioned above, when $\xi < \lambda$, for the model it is advantageous to be split up into a large number of microscopic regions with a characteristic size of the order of $\xi$. The two types of regions are most likely: layers of small thickness and threads of small diameter. Theoretical calculations show that a filamentary structure has less energy. For the first time such structures were discussed by Onsager and Feynman in connection with the phenomenon of superfluidity of helium. In 1956, Abrikosov generalized this approach to the case of superconductivity.

The structure of an isolated vortex filament is shown in Figure 3.11. The thread has a rigid skeleton of the radius $\xi$ in which the density of superconducting electrons decreases to zero at approaching the center. The magnetic field lines are not only in the core region. The field is maximum on the axis of the thread and extends from it to a distance of the order of $\lambda$. We will show that the value of the magnetic flux associated with one thread is equal to one flux quantum (see also section 1.9).

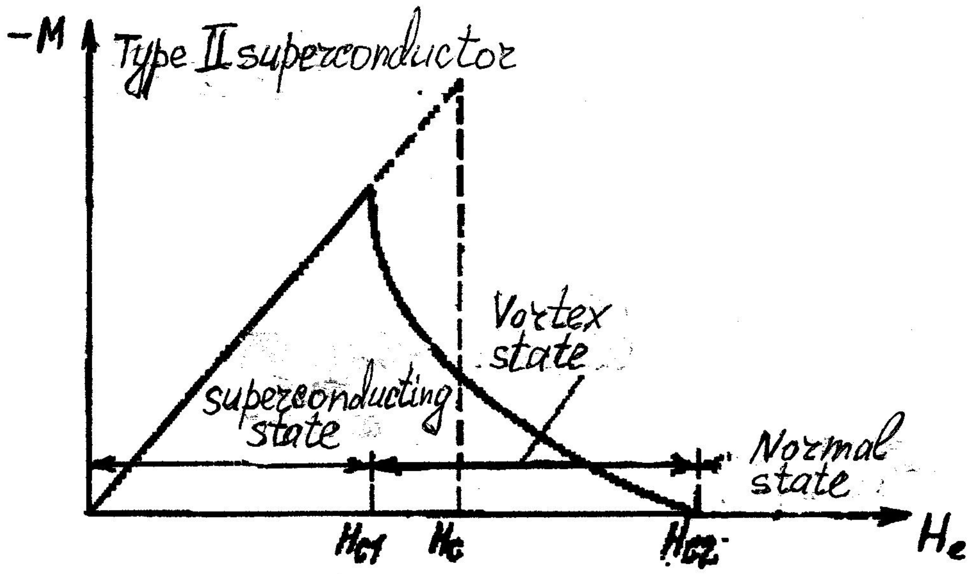


*Figure 3.10. Reversible magnetization curves of type I and II superconductors having the form of a long cylinder.*

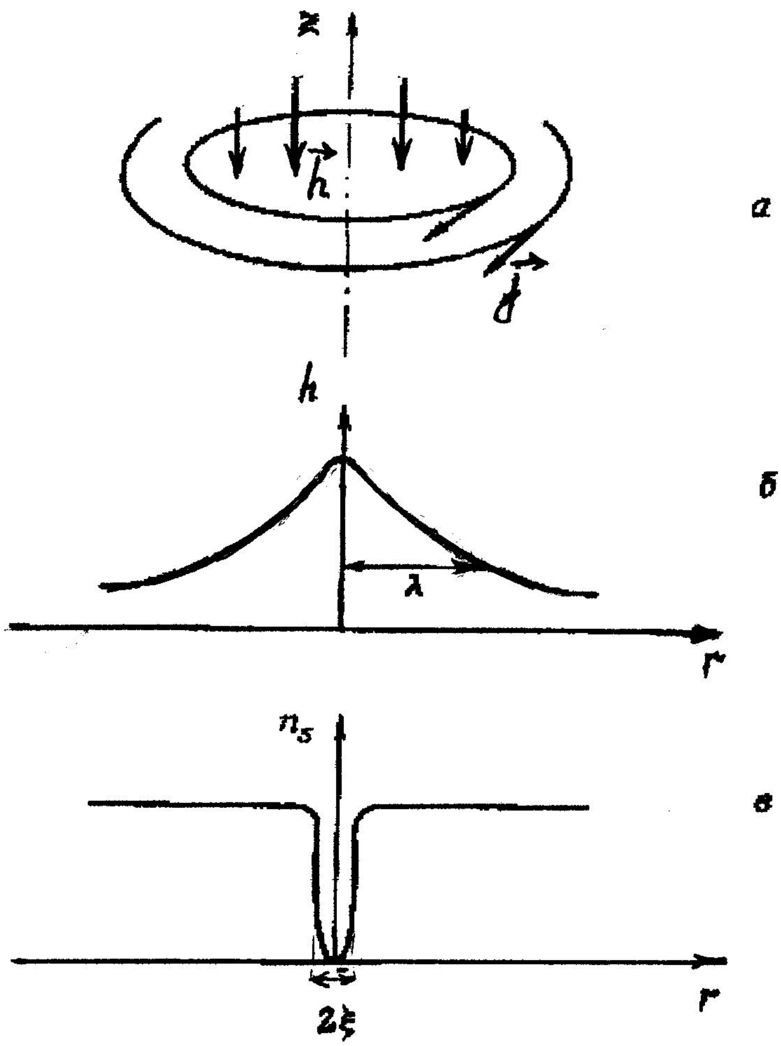


*Figure 3.11. The structure of an isolated vortex filament: a – the configuration of the field and the currents, b – the dependence of the magnetic field on the distance to the filament axis, c - the density of Cooper pairs in the core region of the thread.*

The equation for the current density has a form

$$\vec{j}_S = \frac{e'i\hbar}{2m'}(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi) - \frac{e'^2}{m'}|\psi|^2\vec{A} \qquad (3.15)$$

It should be noted once again that, although (3.15) is one of the Ginzburg-Landau equations (see. (2.43)), its applicability is not limited by the proximity of the temperature to $T_C$, because, as mentioned earlier, it is a general expression for the current density in the quantum mechanics.

Substituting $\psi(\vec{r}) = |\psi(\vec{r})| \cdot \exp(i\varphi(\vec{r}))$ into (3.15), we obtain

$$\vec{j} = \frac{e'}{m'}|\psi|^2(\hbar\vec{\nabla}\varphi - e'\vec{A}) \qquad (3.16)$$

Let us integrate (3.16) over a circle around the axis of the vortex with a radius $r >> \lambda$. At these distances the current density $j$ can be considered as equal to zero. Taking into account that, due to the axial symmetry, the module of the current density is the same at all points of the circle. Therefore, we obtain

$$\hbar[\varphi] = e'\oint\vec{A}\cdot d\vec{l} = e'\int rot\vec{A}\cdot d\vec{S} = e'\int\vec{B}\cdot d\vec{S} = e'\Phi \qquad (3.17)$$

where $[\varphi]$ is the whole phase change, $\Phi$ - the magnetic flux through the loop. From the ambiguity of the function $\psi$ we can infer that $[\varphi] = 2\pi n$, where $n$ is an arbitrary integer. The minimum value of $n$ is 1, so the minimum magnetic flux associated with the thread is equal to one quantum of magnetic flux $\Phi_0 = h/2e = 2\cdot10^{-15}$ Wb. In principle, the vortex can contain any integer number of quanta $\Phi_0$, but for reasons of minimum energy it prefers to disintegrate into several vortices with one quantum $\Phi_0$ each. Indeed, we will see that the energy of the vortex is proportional to the square of the magnetic flux. This means that the energy of a vortex with $n$ quantum $\Phi_0$ is $n$ times more than the energy of $n$ vortices with one $\Phi_0$ each.

**3.3.1. The properties of the isolated vortex filament**

Let us consider in detail the structure of a single vortex filament in the case $\xi << \lambda$.

Since in this case the core of the vortex is small, when calculating the energy we ignore for a time its input, i.e. the change in the energy of the superconducting phase condensation. Then the free energy per unit length of a separate thread is equal

$$F = \frac{\mu_0}{2}\int(h^2 + \lambda^2\left|rot\vec{h}\right|^2)dV \qquad (3.18)$$

where the integral is taken over the area $r > \xi$.

From the condition of minimum $F$ we obtain (beyond the core) the London equation

$$\vec{h} + \lambda^2 rot\,rot\,\vec{h} = 0 \qquad (3.19)$$

Inside the core, strictly speaking, it would be necessary to apply a more complicated equation, but, since the radius of the core is small, we can replace the existing feature by the two-dimensional δ-function:

$$\vec{h} + \lambda^2\, rot\,rot\,\vec{h} = \frac{\vec{\Omega}}{\mu_0}\delta(\vec{r}) \qquad (3.20)$$

where $\vec{\Omega}$ - a vector directed along the thread.

We will show that the modulus of the vector $\vec{\Omega}$ is equal to the flux quantum $\Phi_0$. Integrating (3.20) over the area of the circular path of radius $r$ centered on the axis:

$$\int \vec{h} d\vec{\sigma} + \lambda^2 \oint rot\,\vec{h} \cdot d\vec{l} = \Omega / \mu_0 \qquad (3.21)$$

If the radius of the selected circle is much larger than $\lambda$ ($r >> \lambda$), the currents along the path, and hence all the contour integrals can be neglected. Then we find that the modulus of the vector $\vec{\Omega}$ is equal to the magnetic flux associated with the thread, i.e. $\Omega = \Phi_0$.

Let us solve the equation (3.20), together with the Maxwell equation $div\vec{h} = 0$.

It is easy to find the value of the current density $\vec{j} = rot\,\vec{h}$ in the area $\xi < r << \lambda$. If the integration path lies in this region, then in the equation (3.21) the first term can be neglected, since only a small part of the total flow $\Phi_0$ passes through the circuit. Then we obtain $\lambda^2 2\pi r \left| rot\vec{h} \right| = \Phi_0 / \mu_0$ or

$$j = \left| rot\vec{h} \right| = \frac{\Phi_0}{2\pi\lambda^2 \mu_0} \frac{1}{r}, \text{ at } \xi < r << \lambda \qquad (3.22)$$

Considering $\left| rot\,\vec{h} \right| = -\frac{\partial h}{\partial r}$ and integrating, we obtain

$$h = \frac{\Phi_0}{2\pi\lambda^2 \mu_0} \left( \ln\frac{\lambda}{r} + const \right), \text{ at } \xi < r << \lambda \qquad (3.23)$$

To calculate the constant let us find the exact solution of (3.20). In cylindrical coordinates, this equation has the form (for $r > 0$)

$$h'' + \frac{1}{r} h' + \frac{h}{\lambda^2} = 0 \qquad (3.24)$$

The solution of this equation decreasing when $r \to \infty$ has a form

$$h = C \cdot K_0\left( \frac{r}{\lambda} \right), \text{ at } \xi < r \qquad (3.25)$$

where $K_0$ is the Bessel (Hankel) function of zero order of imaginary argument.

Coefficient $C$ in (3.25) and $const$ in (3.23) can be found of crosslinking (3.23) and (3.25), resulting in (3.23) and (3.25) take the form

$$h = \frac{\Phi_0}{2\pi\lambda^2 \mu_0} \ln\frac{\lambda}{r} \qquad \xi < r << \lambda \qquad (3.26)$$

$$h = \frac{\Phi_0}{2\pi\lambda^2 \mu_0} K_0\left( \frac{r}{\lambda} \right) \qquad \xi < r \qquad (3.27)$$

The asymptotic solution of (3.27) at $r >> \lambda$ takes the form

$$h = \frac{\Phi_0}{2\pi\lambda^2\mu_0}\sqrt{\frac{\pi\lambda}{2r}}\cdot\exp\left(-\frac{r}{\lambda}\right) \qquad (3.28)$$

Knowing the expression for the field, we can find the energy per unit length of the filament $F$. Using the formula of vector analysis $div(\vec{a}\times\vec{b}) = \vec{b}\cdot rot\vec{a} - \vec{a}\cdot rot\vec{b}$ and equation (3.19), we obtain the expression

$$div(\vec{h}\times rot\vec{h}) = (rot\vec{h})^2 - \vec{h}\cdot rot\,rot\vec{h} = (rot\vec{h})^2 + h^2/\lambda^2\,. \qquad (3.29)$$

Substituting (3.29) into (3.18) and applying the Gauss theorem, we obtain

$$F = \frac{\mu_0\lambda^2}{2}\int div(\vec{h}\times rot\vec{h})dV = \frac{\mu_0\lambda^2}{2}\int(\vec{h}\times rot\,\vec{h})\,d\vec{\sigma}\,, \qquad (3.30)$$

where the integration in the last integral is taken over the surface of the core, i.e. of the cylinder with a radius $\xi$. Since on the core surface the vectors $\vec{h}$, $rot\vec{h}$ and $d\vec{\sigma}$ are mutually perpendicular, and their modules, according to (3.22) and (3.26), are constant, they can be taken out from the integral in (3.30). Then we get

$$F = \frac{\mu_0\lambda^2}{2}h(\xi)j(\xi)\cdot 2\pi\xi = \frac{\Phi_0^2}{4\pi\lambda^2\mu_0}\ln\frac{\lambda}{\xi}\,. \qquad (3.31)$$

The accounting of the core energy gives the final expression for the energy per unit length of the thread

$$F = \frac{\Phi_0^2}{4\pi\lambda^2\mu_0}\left(\ln\frac{\lambda}{\xi} + \varepsilon\right), \qquad (3.32)$$

where $\varepsilon \approx 0{,}1$.

**3.3.2. Interaction of vortex filaments**

Let us consider two threads parallel to the z axis and passing at z = 0 through the points $\vec{r}_1 = (x_1, y_1)$ и $\vec{r}_2 = (x_2, y_2)$. The resulting magnetic field distribution is described by the equation

$$\vec{h} + \lambda^2\,rot\,rot\,\vec{h} = \frac{\vec{\Phi}_0}{\mu_0}\left[\delta(\vec{r}-\vec{r}_1) + \delta(\vec{r}-\vec{r}_2)\right], \qquad (3.33)$$

which is a generalization of (3.20).

The solution is a superposition of the two fields $\vec{h}(\vec{r}) = \vec{h}_1(\vec{r}) + \vec{h}_2(\vec{r})$ generated by each of the threads individually

$$\vec{h}_i(\vec{r}) = \frac{\vec{\Phi}_0}{2\pi\lambda^2\mu_0}K_0\left(\frac{|\vec{r}-\vec{r}_i|}{\lambda}\right) \qquad (3.34)$$

To find the free energy of the system it is necessary to calculate the integral (3.30) over the surface of the cores of both vortices that is not as easy as it was for a single thread. To calculate the energy of interacting of two vortices per unit length we should deduct of the energy of the system the own energy of the threads (3.31) that in the case $\xi << \lambda$ gives the following expression

$$U_{12} = \frac{\mu_0 \lambda^2}{2} \int (\vec{h} \times rot\, \vec{h}) d\vec{\sigma} - 2F = \frac{\Phi_0^2}{2\pi \lambda^2 \mu_0} K_0 \left( \frac{|\vec{r}_2 - \vec{r}_1|}{\lambda} \right) > 0 \quad (3.35)$$

The positive interaction energy $U_{12}$ corresponds to the mutual repulsing of the threads. For large distances $r_{12}$ between the threads ($r_{12} >> \lambda$) the interaction energy $U_{12}$ decreases as $\sqrt{\frac{1}{r_{12}}} \cdot \exp\left( -\frac{r_{12}}{\lambda} \right)$.